MATHEMATICS ADMISSIONS TEST

For candidates applying for Mathematics, Computer Science or one of their joint degrees at OXFORD UNIVERSITY and/or IMPERIAL COLLEGE LONDON

Wednesday 5 November 2014

Time Allowed: 2½ hours

Please complete the following details in BLOCK CAPITALS

Surname	
Other names	

Candidate Number	**M**					

This paper contains 7 questions of which you should attempt 5. There are directions throughout the paper as to which questions are appropriate for your course.

A: Oxford Applicants: if you are applying to Oxford for the degree course:

- Mathematics *or* Mathematics & Philosophy *or* Mathematics & Statistics, you should attempt Questions **1,2,3,4,5.**
- Mathematics & Computer Science, you should attempt Questions **1,2,3,5,6.**
- Computer Science *or* Computer Science & Philosophy, you should attempt **1,2,5,6,7.**

Directions under A take priority over any directions in B which are relevant to you.

B: Imperial Applicants: if you are applying to Imperial College for any of the Mathematics courses**:** Mathematics, Mathematics (Pure Mathematics), Mathematics with a Year in Europe, Mathematics with Applied Mathematics/Mathematical Physics, Mathematics with Mathematical Computation, Mathematics with Statistics, Mathematics with Statistics for Finance, Mathematics Optimisation and Statistics, you should attempt Questions **1,2,3,4,5.**

Further credit cannot be obtained by attempting extra questions.

Question 1 is a multiple choice question with ten parts. Marks are given solely for correct answers but any rough working should be shown in the space between parts. Answer Question 1 on the grid on Page 2. Each part is worth 4 marks.

Answers to questions 2-7 should be written in the space provided, continuing on to the blank pages at the end of this booklet if necessary. Each of Questions 2-7 is worth 15 marks.

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Time Allowed: 2½ hours

Please complete these details below in block capitals.

Centre Number														
Candidate Number	**M**													
UCAS Number (if known)				–				–						

	d	d		m	m		y	y
Date of Birth			–			–		

Please tick the appropriate box:

- ☐ I have attempted Questions **1,2,3,4,5**
- ☐ I have attempted Questions **1,2,3,5,6**
- ☐ I have attempted Questions **1,2,5,6,7**

Admissions Testing Service

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Q1	Q2	Q3	Q4	Q5	Q6	Q7

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1. For **ALL APPLICANTS.**

For each part of the question on pages 3-7 you will be given **five** possible answers, just one of which is correct. Indicate for each part **A-J** which answer (a), (b), (c), (d), or (e) you think is correct with a tick (✓) in the corresponding column in the table below. *Please show any rough working in the space provided between the parts.*

	(a)	(b)	(c)	(d)	(e)
A					
B					
C					
D					
E					
F					
G					
H					
I					
J					

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A. The inequality

$$x^4 < 8x^2 + 9$$

is satisfied precisely when

(a) $-3 < x < 3$; (b) $0 < x < 4$; (c) $1 < x < 3$; (d) $-1 < x < 9$; (e) $-3 < x < -1$.

$x^4 < 8x^2 + 9$

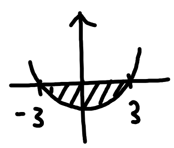
$x^4 - 8x^2 - 9 < 0$

$(x^2 - 9)(x^2 + 1) < 0$

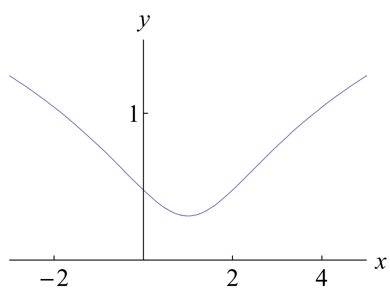
$x^2 + 1 > 0$

$x^2 - 9 < 0$

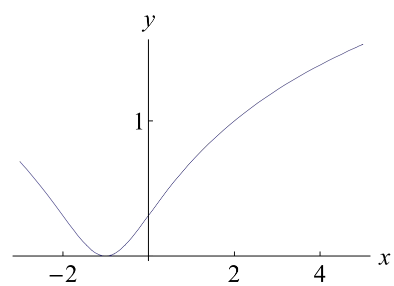
$-3 < x < 3$



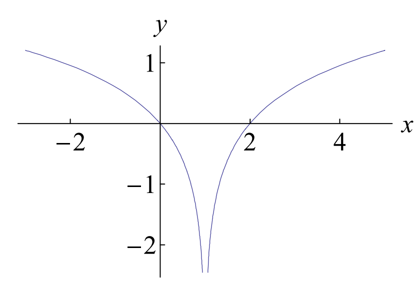
B. The graph of the function $y = \log_{10}(x^2 - 2x + 2)$ is sketched in



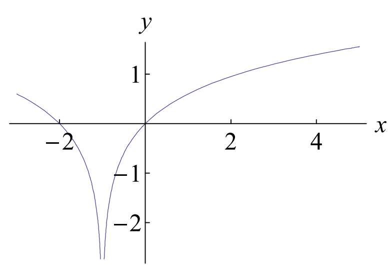
(a)



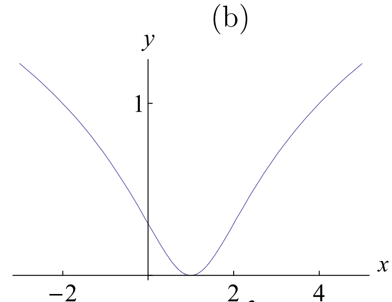
(b)



(c)



(d)



(e)

$y = \log_{10}(x^2 - 2x + 2)$

$10^y = x^2 - 2x + 2$

$x = 0,\ y = ?$

$10^y = 2$

$0 < y < 1$

$(10^0 = 1,\ 10^1 = 10)$

eliminating c and d

$y = 0,\ x = ?$

$1 = x^2 - 2x + 2$

$0 = x^2 - 2x + 1$

$0 = (x-1)^2$

$x = 1$

eliminating a and b

Turn over

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C. The cubic

$$y = kx^3 - (k+1)x^2 + (2-k)x - k$$

has a turning point, that is a minimum, when $x = 1$ precisely for

(a) $k > 0$, (b) $0 < k < 1$, (c) $k > \frac{1}{2}$, (d) $k < 3$, (e) all values of k.

$y = kx^3 - (k+1)x^2 + (2-k)x - k$

$\frac{dy}{dx} = 3kx^2 - 2(k+1)x + (2-k)$

$x = 1 \quad \frac{dy}{dx} = 0$

when $\frac{d^2y}{dx^2} > 0$, turning point is a minimum

$\frac{d^2y}{dx^2} = 6kx - 2(k+1) > 0$

$x = 1$

$6k - 2k - 2 > 0$

$4k > 2$

$k > \frac{1}{2}$

D. The reflection of the point $(1, 0)$ in the line $y = mx$ has coordinates

(a) $\left(\frac{m^2+1}{m^2-1}, \frac{m}{m^2-1}\right)$, (b) $(1, m)$, (c) $(1-m, m)$,

(d) $\left(\frac{1-m^2}{1+m^2}, \frac{2m}{1+m^2}\right)$, (e) $(1-m^2, m)$.

distance of $(1,0)$ and its reflection in $y=mx$ from a point of $y=mx$ would be equal

$(0,0)$ is a point on $y=mx$

Distance of $(1,0)$ from $(0,0)$ is 1

Distance from $(0,0)$

a) $\sqrt{\left(\frac{m^2+1}{m^2-1}\right)^2 + \left(\frac{m}{m^2-1}\right)^2}$

$= \sqrt{\frac{m^4 + 2m^2 + 1 + m^2}{(m^2-1)^2}} \neq 1$

b) $\sqrt{1+m^2} \neq 1$

c) $\sqrt{(1-m)^2 + m^2} = \sqrt{1 + m^2 - 2m + m^2}$

$\neq 1$

d) $\sqrt{\left(\frac{1-m^2}{(1+m^2)}\right)^2 + \left(\frac{2m}{1+m^2}\right)^2}$

$= \sqrt{\frac{1 + m^4 - 2m^2 + 4m^2}{(1+m^2)^2}}$

$= \sqrt{\frac{(1+m^2)^2}{(1+m^2)^2}} = 1$

e) $\sqrt{(1-m^2)^2 + m^2} = \sqrt{1 + m^4 - 2m^2 + m^2}$

$\neq 1$

E. As x varies over the real numbers, the largest value taken by the function

$$\left(4\sin^2 x + 4\cos x + 1\right)^2$$

equals

(a) $17+12\sqrt{2}$, (b) 36, (c) $48\sqrt{2}$, (d) $64-12\sqrt{3}$, (e) 81.

$\cos^2 x + \sin^2 x = 1$

$(4\sin^2 x + 4\cos x + 1)^2 = (4(1-\cos^2 x) + 4\cos x + 1)^2$

$= (5 - 4\cos^2 x + 4\cos x)^2$

$= (-4(\cos^2 x - \cos x - \frac{5}{4}))^2$

$= (-4((\cos x - \frac{1}{2})^2 - \frac{1}{4} - \frac{5}{4}))^2$

$= (6 - 4(\cos x - \frac{1}{2})^2)^2$

$-4(\cos x - \frac{1}{2})^2 < -12$

$(\cos x - \frac{1}{2})^2 > 3$

$\cos x > \pm\sqrt{3} + \frac{1}{2}$

$-1 \leq \cos x \leq 1$

$(\cos x - \frac{1}{2})^2 > 0$

$\therefore$ max value when $(\cos x - \frac{1}{2}) = 0$

$= (6-0)^2 = 36$

F. The functions S and T are defined for real numbers by

$$S(x) = x + 1, \qquad \text{and} \qquad T(x) = -x.$$

The function S is applied s times and the function T is applied t times, *in some order*, to produce the function

$$F(x) = 8 - x.$$

It is possible to deduce that:

(a) $s = 8$ and $t = 1$.
(b) s is odd and t is even.
(c) s is even and t is odd.
(d) s and t are powers of 2.
(e) none of the above.

S adds 1 to a function. If a number is odd and the sign is reversed by T, the number will remain odd (and likewise with even numbers). Therefore, in order to achieve the 8 in the function F, the applications of S required will always be even. 18 is an even number.

T changes the sign of x. As the sign of x in F is negative, the applications of T required will always be odd.

(By applying S once, T once and S a further 8 times, resulting in $s=9$, $t=1$, a, b and d are eliminated)

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G. Let n be a positive integer. The coefficient of x^3y^5 in the expansion of

$$(1 + xy + y^2)^n$$

equals

(a) n, (b) 2^n, (c) $\binom{n}{3}\binom{n}{5}$, (d) $4\binom{n}{4}$, (e) $\binom{n}{8}$.

x^3y^5 is formed by $xy \times xy \times xy \times y^2$ from the expansion of $(1+xy+y^2)^n$. This can be writen as $(xy^3) \times (y^2)^1$ and can be written in 4 ways of different arrangement.

$3+1=4$, resulting in $\binom{n}{4}$

$\therefore$ the coefficient is $4\binom{n}{4}$ and the answer is d.

H. The function $F(n)$ is defined for all positive integers as follows: $F(1) = 0$ and for all $n \geqslant 2$,

$$\begin{aligned} F(n) &= F(n-1) + 2 && \text{if 2 divides } n \text{ but 3 does not divide } n; \\ F(n) &= F(n-1) + 3 && \text{if 3 divides } n \text{ but 2 does not divide } n; \\ F(n) &= F(n-1) + 4 && \text{if 2 and 3 both divide } n; \\ F(n) &= F(n-1) && \text{if neither 2 nor 3 divides } n. \end{aligned}$$

The value of $F(6000)$ equals

(a) 9827, (b) 10121, (c) 11000, (d) 12300, (e) 12352.

Between 1-6000, 3000 values of n are divisible by 2 and 2000 by 3

Even multiples of 3 are divisible by 2 (1000 values)

$\therefore f(6000) = 2 \times (3000-1000) + 3(2000-1000) + 4(1000)$

$= 11000$

I. The graph of the function
$$y = 2^{x^2-4x+3}$$
can be obtained from the graph of $y = 2^{x^2}$ by

(a) a stretch parallel to the y-axis followed by a translation parallel to the y-axis.
(b) a translation parallel to the x-axis followed by a stretch parallel to the y-axis.
(c) a translation parallel to the x-axis followed by a stretch parallel to the x-axis.
(d) a translation parallel to the x-axis followed by reflection in the y-axis.
(e) reflection in the y-axis followed by translation parallel to the y-axis.

$y = 2^{x^2-4x+3}$
$= 2^{(x-2)^2-4+3}$
$= 2^{(x-2)^2-1}$
$= 2^{(x-2)^2} \times 2^{-1}$
$= \frac{1}{2} \times 2^{(x-2)^2}$

Translation parallel to x axis
(2 units to the right)

Stretch parallel to y axis
(scale factor $\frac{1}{2}$)

J. For all real numbers x, the function $f(x)$ satisfies
$$6 + f(x) = 2f(-x) + 3x^2 \left(\int_{-1}^{1} f(t)\,\mathrm{d}t \right).$$
It follows that $\int_{-1}^{1} f(x)\,\mathrm{d}x$ equals

(a) 4, (b) 6, (c) 11, (d) $\frac{27}{2}$, (e) 23.

$6 + f(x) = 2f(-x) + 3x^2 \int_{-1}^{1} f(t)\,dt$

$\int_{-1}^{1} 6\,dx + \int_{-1}^{1} f(x)dx = 2\int_{-1}^{1} f(-x)\,dx + \int_{-1}^{1} 3x^2\,dx \times \int_{-1}^{1} f(t)\,dt$

$\int_{-1}^{1} f(t)\,dt$ is a constant equal to $\int_{-1}^{1} f(x)\,dx$

$\int_{-1}^{1} f(-x)\,dx = \int_{-1}^{1} f(x)\,dx$, as $\int_{-1}^{1} f(-x)\,dx$ represents the area under the graph of the reflection in the y axis of $\int_{-1}^{1} f(x)\,dx$, which is the same

$[6x]_{-1}^{1} + \int_{-1}^{1} f(x)\,dx = 2\int_{-1}^{1} f(x)\,dx + [x^3]_{-1}^{1} \times \int_{-1}^{1} f(x)\,dx$

$6-(-6) = 3\int_{-1}^{1} f(x)\,dx = 12$

$\int_{-1}^{1} f(x)\,dx = 4$

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2. For **ALL APPLICANTS.**

Let a and b be real numbers. Consider the cubic equation

$$x^3 + 2bx^2 - a^2x - b^2 = 0. \qquad (*)$$

(i) Show that if $x = 1$ is a solution of $(*)$ then

$$1 - \sqrt{2} \leqslant b \leqslant 1 + \sqrt{2}.$$

(ii) Show that there is no value of b for which $x = 1$ is a repeated root of $(*)$.

(iii) Given that $x = 1$ is a solution, find the value of b for which $(*)$ has a repeated root.

For this value of b, does the cubic

$$y = x^3 + 2bx^2 - a^2x - b^2$$

have a maximum or minimum at its repeated root?

i) $x^3 + 2bx^2 - a^2x - b^2 = 0$

$x = 1$

$1 + 2b - a^2 - b^2 = 0$

$b^2 - 2b - 1 + a^2 = 0$

$(b-1)^2 - 2 + a^2 = 0$

$(b-1)^2 = 2 - a^2$

$(b-1)^2 \geqslant 0$, $a^2 \geqslant 0$

$0 \leqslant (b-1)^2 \leqslant 2$

$-\sqrt{2} \leqslant b - 1 \leqslant \sqrt{2}$

$\sqrt{2} + 1 \leqslant b \leqslant \sqrt{2} + 1$

ii) from above $a^2 = 1 + 2b - b^2$

$\therefore x^3 + 2bx^2 + (b^2 - 2b - 1)x - b^2 = 0$

factorising by $(x-1)$

$\Rightarrow (x-1)(x^2 + (2b+1)x + b^2)$

true because if multiplied out it equals line 2

if $x-1$ is repeated

1 is a root of $x^2 + (2b+1)x + b^2$

$1 + 2b + 1 + b^2 = 0$

$b^2 + 2b + 2 = 0$

$2^2 - 4 \times 2 < 0$

$\therefore$ discriminant < 0

$\therefore$ no solutions where $x-1$ is a repeated root

iii) from part ii : $x^3 + 2bx^2 + (b^2 - 2b - 1)x - b^2 = 0$

which is the same as $(x-1)(x^2 + (2b+1)x + b^2)$

if a repeated root it is not 1 (from part ii)

$\therefore x^2 + (2b+1)x + b^2 = 0$ has a repeated root

$\therefore (2b+1)^2 - 4 \times b^2 = 0$

$4b^2 + 4b + 1 - 4b^2 = 0$

$4b + 1 = 0 \quad \therefore b = -\frac{1}{4}$

$y = x^3 + 2bx^2 - a^2x - b^2$

$\frac{dy}{dx} = 3x^2 + 4bx - a^2$

$\frac{d^2y}{dx^2} = 6x + 4b \qquad x = b = -\frac{1}{4}$

$\frac{d^2y}{dx^2} = \frac{-6}{4} - 1 = \frac{-5}{2} < 0$

$\therefore$ maximum at repeated root

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3.

For **APPLICANTS IN** $\left\{ \begin{array}{l} \textbf{MATHEMATICS} \\ \textbf{MATHEMATICS \& STATISTICS} \\ \textbf{MATHEMATICS \& PHILOSOPHY} \\ \textbf{MATHEMATICS \& COMPUTER SCIENCE} \end{array} \right\}$ **ONLY.**

Computer Science and *Computer Science & Philosophy* **applicants should turn to page 14.**

The function $f(x)$ is defined for all real numbers and has the following properties, valid for all x and y:

(A) $f(x+y) = f(x)\, f(y)$.
(B) $\mathrm{d}f/\mathrm{d}x = f(x)$.
(C) $f(x) > 0$.

Throughout this question, these should be the only properties of f that you use; no marks will be awarded for any use of the exponential function.

Let $a = f(1)$.

(i) Show that $f(0) = 1$.

(ii) Let

$$I = \int_0^1 f(x)\,\mathrm{d}x.$$

Show that $I = a - 1$.

(iii) The trapezium rule with n steps is used to produce an estimate I_n for the integral I. Show that

$$I_n = \frac{1}{2n}\left(\frac{b+1}{b-1}\right)(a-1)$$

where $b = f(1/n)$.

(iv) Given that $I_n \geqslant I$ for all n, show that

$$a \leqslant \left(1 + \frac{2}{2n-1}\right)^n.$$

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3i. $f(1+0) = f(0) \times f(1)$

$f(1) = f(0) \times f(1)$

$f(0) = 1$

ii. $I = \int_0^1 f(x)\,dx \qquad f(x) = \frac{df}{dx}$

$I = \int_0^1 \frac{df}{dx}\,dx$

$I = \left[f(x)\right]_0^1$

$I = f(1) - f(0)$

$I = a - 1$

iii. $I_n = \frac{1}{2} \times \frac{1}{n}\left[1 + a + 2\left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \dots f\left(\frac{n-1}{n}\right)\right)\right]$

$f\left(\frac{1}{n}\right) = b \qquad f\left(\frac{1}{n} + \frac{1}{n}\right) = f\left(\frac{2}{n}\right) = f\left(\frac{1}{n}\right) \times f\left(\frac{1}{n}\right) = b^2$

$f\left(\frac{k}{n}\right) = b^k$, where $k \geq 0$

$$I_n = \frac{1}{2n}\left[1 + a + 2\underbrace{\left(b + b^2 + \dots + b^{n-1}\right)}_{\text{Geometric series}}\right]$$

$b + b^2 + \dots + b^{n-1} = \dfrac{b(b^{n-1} - 1)}{b - 1} \qquad a = f(1) = f\left(\frac{n}{n}\right) = b^n$

$I_n = \frac{1}{2n}\left[1 + b^n + \frac{2b^n - 2b}{b - 1}\right]$

$I_n = \frac{1}{2n}\left[\frac{b + b^{n+1} - 1 - b^n + 2b^n - 2b}{b - 1}\right]$

3iii (continued)

$$I_n = \frac{1}{2n}\left[\frac{b^{n+1}+b^n-b-1}{b-1}\right]$$

$$I_n = \frac{1}{2n}\left[\frac{(b+1)(b^n-1)}{b-1}\right] \qquad b^n = a$$

$$I_n = \frac{1}{2n}\left(\frac{b+1}{b-1}\right)(a-1)$$

iv. $\frac{1}{2n}\left(\frac{b+1}{b-1}\right)(a-1) \geq (a-1)$ $\qquad n \neq 0 \quad b^n > 1$

$a > 1$

$(a-1) \neq 0$

$$\frac{1}{2n}\left(\frac{b+1}{b-1}\right) \geq 1$$

$$b+1 \geq 2n(b-1)$$

$$b+1 \geq 2nb-2n$$

$$1+2n \geq b(2n-1)$$

$$b \leq \left(\frac{1+2n}{2n-1}\right)$$

$$b^n \leq \left(\frac{2+2n-1}{2n-1}\right)^n \qquad b^n = a$$

$$a \leq \left(1 + \frac{2}{2n-1}\right)^n$$

hence $a^{\frac{1}{n}} \leq 1 + \frac{2}{2n-1}$

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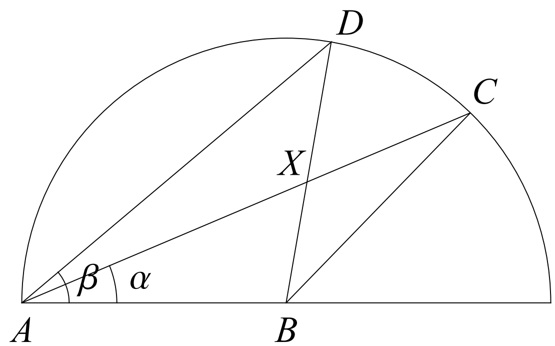
4.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**

Mathematics & Computer Science, *Computer Science* and *Computer Science & Philosophy* **applicants should turn to page 14.**

In the diagram below is sketched a semicircle with centre B and radius 1. Three points A, C, D lie on the semicircle as shown with α denoting angle CAB and β denoting angle DAB. The triangles ABC and ABD intersect in a triangle ABX.

Throughout the question we shall consider the value of α fixed. Assume for now that $0 < \alpha \leqslant \beta \leqslant \pi/2$.



(i) Show that the area of the triangle ABC equals

$$\frac{1}{2}\sin(2\alpha).$$

(ii) Let

$$F = \frac{\text{area of triangle } ABX}{\text{area of triangle } ABC}.$$

Without calculation, explain why, for every k in the range $0 \leqslant k \leqslant 1$, there is a unique value of β such that $F = k$.

(iii) Find the value of β such that $F = 1/2$.

(iv) Show that

$$F = \frac{\sin(2\beta)\sin\alpha}{\sin(2\beta - \alpha)\sin(2\alpha)}.$$

(v) Suppose now that $0 < \beta < \alpha \leqslant \pi/2$. Write down, without further calculation, an expression for the area of ABX and hence a formula for F.

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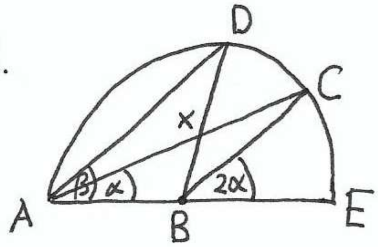
4i.



$\angle CAE \times 2 = \angle CBE$

$\angle CBE = 2\alpha$

Area of triangle $= \frac{1}{2} \times AB \times BC \times \sin(\pi - 2\alpha)$

$AB = BC = \text{radius} = 1$

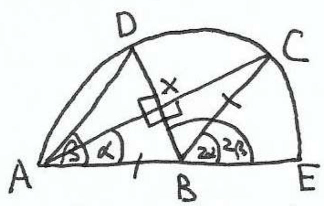
$\sin(\pi - 2\alpha) = \sin(2\alpha)$

$\text{Area} = \frac{1}{2} \times 1 \times 1 \times \sin(2\alpha)$

$= \frac{1}{2}\sin(2\alpha)$

ii. $B = \frac{\pi}{2}$, **F=0** $B = \alpha$, **F=1** As β increases, F decreases. $\therefore$ for every k in the range $0 \leq k \leq 1$, there is a unique value of β such that $F = k$

iii.



When $\angle AXB = 90°$, Area of ABX $=$ Area of XCB and $F = \frac{1}{2}$

$\angle DAE \times 2 = \angle DBE$

$\angle DBE = 2\beta$

$\therefore \angle DBA = \pi - 2\beta$

$\angle DBA = \pi - \left(\frac{\pi}{2} + \alpha\right)$

$= \frac{\pi}{2} - \alpha$

$\frac{\pi}{2} - \alpha = \pi - 2\beta \qquad 2\beta = \frac{\pi}{2} + \alpha \qquad \beta = \frac{\pi}{4} + \frac{\alpha}{2}$

iv. $\angle AXB = \pi - (\pi - 2\beta + \alpha)$

$= 2\beta - \alpha$

Sine rule

$\frac{AB}{\sin(2\beta - \alpha)} = \frac{BX}{\sin\alpha} \qquad AB = 1$

$BX = \frac{\sin\alpha}{\sin(2\beta - \alpha)}$

Area of ABX $= \frac{1}{2} \times BX \times AB \times \sin(\pi - 2\beta)$

$= \frac{1}{2} \times \frac{\sin\alpha}{\sin(2\beta - \alpha)} \times 1 \times \sin(2\beta)$

$= \frac{\sin\alpha \sin(2\beta)}{2\sin(2\beta - \alpha)}$

$F = \frac{\sin\alpha \sin(2\beta)}{2\sin(2\beta - \alpha)} \div \frac{1}{2}\sin(2\alpha)$

$F = \frac{\sin\alpha \sin(2\beta)}{\sin(2\beta - \alpha)\sin(2\alpha)}$

v. Area of ABX $= \frac{\sin\beta \sin 2\alpha}{2\sin(2\alpha - \beta)}$

$F = \frac{\sin\beta \sin 2\alpha}{2\sin(2\alpha - \beta)} \div \frac{1}{2}\sin(2\alpha) = \frac{\sin\beta}{\sin(2\alpha - \beta)}$

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5. For **ALL APPLICANTS.**

Poets use *rhyme schemes* to describe which lines of a poem rhyme. Each line is denoted by a letter of the alphabet, with the same letter given to two lines that rhyme. To say that a poem has the rhyming scheme ABABCDED, indicates that the first and third lines rhyme, the second and fourth lines rhyme, and the sixth and eighth lines rhyme, but no others.

More precisely, the first line of the poem is given the letter A. If a subsequent line rhymes with an earlier line, it is given the same letter; otherwise, it is given the first unused letter. (For the purposes of this question, you can assume that we have an infinite supply of "letters", not just the 26 letters of the alphabet.)

The purpose of this question is to investigate how many different rhyme schemes there are for poems of n lines. We write r_n for this number.

(i) There are five different rhyming schemes for poems of three lines (so $r_3 = 5$). List them.

Let $c_{n,k}$ denote the number of rhyme schemes for poems with lines n that use exactly k different letters. For example $c_{3,2} = 3$ corresponding to the rhyming schemes AAB, ABA and ABB.

(ii) What is $c_{n,1}$ for $n \geqslant 1$?

What is $c_{n,n}$?

Explain your answers.

(iii) Suppose that $1 < k < n$. By considering the final letter of a rhyming scheme, explain why

$$c_{n,k} = kc_{n-1,k} + c_{n-1,k-1} .$$

(iv) Write down an equation showing how to calculate r_n in terms of the $c_{n,k}$. Hence calculate r_4.

(v) Give a formula for $c_{n,2}$ in terms of n (for $n \geqslant 2$). Justify your answer.

5i. AAA, AAB, ABA, ABB, ABC

ii. $C_{n,1} = 1$ There is one letter for n lines, making $C_{n,1}$ A repeated n times, which only has one possible arrangement.

$C_{n,n} = 1$ There are n different letters with no rhymes. There is only one arrangement where is the same number of letters as lines.

iii. If the last symbol is the same as an earlier symbol: the number of arrangements for the first n-1 lines is $C_{n-1,k}$. There are k possibilities for the last symbol. Therefore number of arrangements is $kC_{n-1,k}$.

If the last symbol is different to earlier symbols: If the last symbol is different to earlier symbols: the number of arrangements for the first n-1 lines is $C_{n-1,k-1}$. There is 1 possibility for the last symbol (kth letter) Therefore number of arrangements is $C_{n-1,k-1}$.

$\therefore$, for both options, $C_{n,k} = kC_{n-1,k} + C_{n-1,k-1}$

iv.

$$r_n = \sum_{k=1}^{n} C_{n,k}$$

$r_4 = C_{4,1} + C_{4,2} + C_{4,3} + C_{4,4}$ $\quad (C_{n,n} = 1, \; C_{n,1} = 1)$

$r_4 = 1 + (2 \times C_{3,2} + C_{3,1}) + (3C_{3,3} + C_{3,2}) + 1$ $\quad (C_{n,k} = kC_{n-1,k} + C_{n-1,k-1})$

$r_4 = 1 + 2 \times 3 + 1 + 3 + 3 + 1$ $\quad (C_{3,2} = 3)$

$r_4 = 1 + 7 + 6 + 1$

$r_4 = 15$

v. $C_{n,2} = 2C_{n-2,2} + 1$

$$\left(\begin{array}{l} \text{sub into } C_{n,k} = kC_{n-1,k} + C_{n-1,k-1} \\ C_{n,2} = 2C_{n-1,2} + C_{n-1,1} \\ C_{n,2} = 2C_{n-1,2} + 1 \end{array} \right)$$

6.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{COMPUTER SCIENCE}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\\ \textbf{COMPUTER SCIENCE \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**

Alice plays a game 5 times with her friends Sam and Pam. In each game Alice chooses two integers x and y with $1 \leqslant x \leqslant y$. She whispers the sum $x + y$ to Sam, and the product $x \times y$ to Pam, so that neither knows what the other was told. Sam and Pam then have to try to work out what the numbers x and y are. Sam and Pam are well known expert logicians.

(i) In the first game, Pam says "I know x and y."

What can we deduce about the values of x and y? Explain your answer.

(ii) In the second game, Pam says "I don't know what x and y are."

Sam then says "I know x and y."

Suppose the sum is 4. What are x and y? Explain your answer.

(iii) In the third game, Pam says "I don't know what x and y are."

Sam then says "I don't know what x and y are."

Pam then says "I now know x and y."

Suppose the product is 4. What are x and y? Explain your answer.

(iv) In the fourth game, Pam says "I don't know what x and y are."

Sam then says "I already knew that."

Pam then says "I now know x and y."

Suppose the product is 8. What are x and y? Explain your answer.

(v) Finally, in the fifth game, Pam says "I don't know what x and y are."

Sam then says "I don't know what x and y are."

Pam then says "I don't know what x and y are."

Sam then says "I now know x and y."

Suppose the sum is 5. What are x and y? Explain your answer.

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6i. $x=1$, $y=1$ or $y=$ prime number

Prime numbers and 1 only have one possible factorisation. All other numbers have more. Since Pam knows immediately, the product must be 1 or a prime number.

ii. $x=2$, $y=2$

If the sum is 4, there are two options - $x=1, y=3$ or $x=2, y=2$. If the former, Pam would know the answer (part i). Since she does not, the latter option is true.

iii. $x=1$, $y=4$

If the product is 4, there are two options: $x=2, y=2$ or $x=1, y=\cancel{3}4$. Pam knows Sam has received the sum of either 4 or 5. If he received 4 and the knowledge that Pam did not know the answer, Sam would know the answer (part ii). Since he does not, Pam knows he must have received the sum as 5 and therefore the second option is true.

iv. $x=1$, $y=8$

If the product is 8, there are two options: $x=2, y=4$ or $x=1, y=8$. Pam knows Sam has received the sum of either 6 or 9. Therefore Pam knows possible options Sam is considering:

Sum: 6		Product
1	5	5
2	4	8
3	3	9

Sum: 9		Product
1	8	8
2	7	14
3	6	18
4	5	20

5 can only be factorised in one way. If Sam received 6, he would not know that Pam was unaware of the answer. All of 8, 14, 18 and 20 can be factorised in more than one way, meaning Sam would know Pam did not know. Therefore Pam knows that the sum of x and y is 9, and $x=1$, $y=8$.

v. $x=2$, $y=3$

If the sum is 5, there are two options: $x=1, y=4$ or $x=2, y=3$. Sam knows the product is either 4 or 6. If 4, when Sam says he does not know x and y, Pam would then know x and y (part iii). Since Pam says she does not know, (for the second time) Sam knows the product must be 6 and $x=2$, $y=3$.

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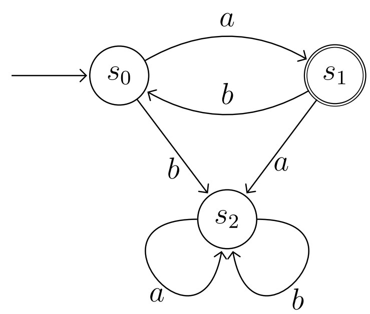
7.

For **APPLICANTS IN** $\left\{ \begin{array}{l} \textbf{COMPUTER SCIENCE} \\ \textbf{COMPUTER SCIENCE \& PHILOSOPHY} \end{array} \right\}$ **ONLY.**

A *finite automaton* is a mathematical model of a simple computing device. A small finite automaton is illustrated below.



A finite automaton has some *finite* number of states; the above automaton has three states, labelled s_0, s_1 and s_2. The initial state, s_0, is indicated with an incoming arrow. The automaton receives *inputs* (e.g. via button presses), which might cause it to change state. In the example, the inputs are a and b. The state changes are illustrated by arrows; for example, if the automaton is in state s_1 and it receives input b, it changes to state s_0; if it is in state s_2 and receives either input, it remains in state s_2. (For each state, there is precisely one out-going arrow for each input.)

Some of the states are defined to be *accepting states*; in the example, just s_1 is defined to be an accepting state, represented by the double circle. A *word* is a sequence of inputs. The automaton *accepts* a word w if that sequence of inputs leads to an accepting state from the initial state. For example, the above automaton accepts the word aba.

(i) Write down a description of the set of words accepted by the above automaton. A clear but informal description will suffice.

(ii) Suppose we alter the above automaton by swapping accepting and non-accepting states; i.e. we make s_0 and s_2 accepting, and make s_1 non-accepting. Write down a description of the set of words accepted by this new automaton. Again, a clear but informal description will suffice.

(iii) Draw an automaton that accepts all words containing an even number (possibly zero) of a's and any number of b's (and no other words).

(iv) Now draw an automaton that accepts all words containing an even number of a's *or* an odd number of b's (and no other words).

Let a^n represent n consecutive a's. Let L be the set of all words of the form $a^n b^n$ where $n = 0, 1, 2, \ldots$; i.e. all words composed of some number of a's followed by the *same* number of b's. We will show that no finite automaton accepts precisely this set of words.

(v) Suppose a particular finite automaton A *does* accept precisely the words in L. Show that if $i \neq j$ then the words a^i and a^j must lead to different states of A.

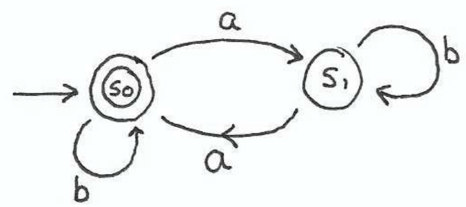
Hence show that this leads to a contradiction.

End of last question

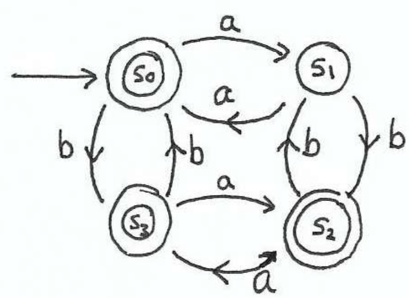
7i. Words beginning and ending with a, with letters alternating between a and b.

ii. Words with only "a"s, only "b"s or both, except those beginning and ending with a, with letters alternating between a and b. (all words except those in part a)

iii.



iv.



v. a^ib^i would lead to an accepting state as it is in the form of a^nb^n $(n=i)$. a^jb^i would not lead to an accepting state as it is not in the form of a^nb^n $(i\neq j)$. Therefore a^i and a^j lead to different states of A. Therefore all $a^1 \rightarrow a^n$ lead to different states of A, indicating an infinite number of states. However, the automaton is finite, so there cannot be an infinite number of states. Therefore, there is no finite automaton that precisely accepts words of the form a^nb^n.